The Algebraic Construction of Integrable Hierarchies, Solitons and Infinite Dimensional Algebras -Lecture 2

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Aim

- Discuss Construction of Non-Linear Differential equations as sinh-Gordon, mKdV, etc and Infinite Dimensional Algebras
- Discuss the general structure of *time evolution integrable* equations associated to *graded Affine Lie algebraic* structure.
- Representation Theory of Infinite Dimensional Algebras and the Systematic Construction of Soliton Solutions.
- Conformal Transformation in 2D and the Virasoro Algebra. String Theory, Critical Exponents in 2D Statistical Models, etc

- Consider two traceless matrices

$$A = \begin{pmatrix} \partial\phi & 1 \\ \lambda & -\partial\phi \end{pmatrix}, \qquad \bar{A} = \begin{pmatrix} 0 & \frac{e^{-2\phi}}{\lambda} \\ e^{2\phi} & 0 \end{pmatrix},$$

- such that

$$\bar{\partial}A - \partial\bar{A} = \bar{\partial}\begin{pmatrix} \partial\phi & 1 \\ \lambda & -\partial\phi \end{pmatrix} - \partial\begin{pmatrix} 0 & \frac{e^{-2\phi}}{\lambda} \\ e^{2\phi} & 0 \end{pmatrix} \tag{1}$$

and

$$[A, \bar{A}] = \begin{pmatrix} e^{2\phi} - e^{-2\phi} & \frac{2\partial\phi e^{-2\phi}}{\lambda} \\ -2\partial\phi e^{2\phi} & e^{2\phi} - e^{-2\phi} \end{pmatrix}$$

- $z = x + t, \quad \bar{z} = x - t; \qquad A = A_x + A_t, \quad \bar{A} = A_x - A_t.$

- It therefore follows that **off diagonal matrix elements cancels and there is no λ dependence**, i.e.,

$$\bar{\partial} A - \partial \bar{A} - [A, \bar{A}] = (\bar{\partial}\partial\phi - e^{2\phi} + e^{-2\phi}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (2)$$

- If ϕ satisfy sinh-Gordon eqn., $\rightarrow$ the *curvature is zero*.
- The solution for zero curvarure is **pure gauge** potentials, i.e.,

$$A_\mu = T^{-1}\partial_\mu T \qquad \text{or} \qquad T = Pe^{\int A_\mu dx^\mu}$$

- **Consequences of zero curvature**
- Integrating in x from $-L \leq x \leq L$ and in t from $0 \leq t \leq \tau$,

$$Pe^{\int_{-L}^{L} A_x(y,t=0)dy} \cdot Pe^{\int_0^{\tau} A_t(y=-L,t)dt},$$

which is equal if we integrate first in t from $0 \leq t \leq \tau$ and then in x from $-L \leq x \leq L$.

$$Pe^{\int_0^{\tau} A_t(y=L,t)dt} \cdot Pe^{\int_{-L}^{L} A_x(y,t=\tau)dy},$$

- Considering periodic boundary conditions, $A_t(y=-L,t)=A_t(y=L,t)$,

$$Tr\left(Pe^{\int_{-L}^{L}A_x(y,t=0)dy}\right)=Tr\left(Pe^{\int_{-L}^{L}A_x(y,t=\tau)dy}\right),$$

- and since A_x depends upon the spectral parameter λ, i.e., $A_x=A_x(y,t,\lambda)$ we can derive the conservation laws,

$$Tr\left(Pe^{\int_{-L}^{L}A_x(y,t,\lambda)dy}\right)=\sum_n \lambda^n Q_n$$

- **Conclusion: Zero curvature representation implies infinite number of conserved charges** and hanceforth Integrability of the model.
- Secret is the **Algebraic Structure** underlying the construction of A and $\bar{A}$.

- Angular Momentum Algebra

$$[h, E_{\pm\alpha}] = \pm 2E_{\pm\alpha}, \qquad [E_\alpha, E_{-\alpha}] = h,$$

where

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E_\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad E_{-\alpha} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

$$A = \begin{pmatrix} \partial\phi & 1 \\ \lambda & -\partial\phi \end{pmatrix} = \partial\phi h + E_\alpha + \lambda E_{-\alpha} \tag{3}$$

- **Diagonal** Field dependent term: $\partial\phi(x, t)h$
- Constant **off Diagonal**, spectral parameter, λ dependent term: $E_\alpha + \lambda E_{-\alpha}$

- Notice that $Q = \frac{1}{2}h \to$ induces a natural gradation that associates grades $\pm 1, 0$ to generators $E_{\pm\alpha}, h$ respectively, i.e.,

$$[Q, E_{\pm\alpha}] = \pm E_{\pm\alpha}, \qquad [Q, h] = 0, \tag{4}$$

- Need systematic picture to include $\lambda \in C$ within algebra $\longrightarrow$ *Affine Algebras*

- Affine Lie Algebra, e.g., $\hat{\mathcal{G}} = \hat{s}l(2) = \{h^{(n)} = \lambda^n h, \quad E_{\pm\alpha}^{(n)} = \lambda^n E_{\pm\alpha}, \; n \in Z\}$

- Loop Algebra $\quad [h^{(m)}, h^{(n)}] = 0,$

$$[h^{(m)}, E_{\pm\alpha}^{(n)}] = \pm 2 E_{\pm\alpha}^{(n+m)}, \qquad [E_{\alpha}^{(m)}, E_{-\alpha}^{(n)}] = h^{(m+n)}$$

- Grading operator, Q, such that $\hat{\mathcal{G}} = \oplus \mathcal{G}_i$ where,

$$[Q, \mathcal{G}_i] = i\mathcal{G}_i \quad \text{and} \quad [\mathcal{G}_i, \mathcal{G}_j] \subset \mathcal{G}_{i+j}$$

e.g., **principal gradation**, $Q = 2\lambda \frac{d}{d\lambda} + \frac{1}{2} h$,

$$\mathcal{G}_{2m} = \{h^{(m)} = \lambda^m h\},$$

$$\mathcal{G}_{2m+1} = \{\lambda^m (E_\alpha + \lambda E_{-\alpha}),\ \lambda^m (E_\alpha - \lambda E_{-\alpha})\}$$

$m = 0, \pm 1, \pm 2, \ldots$.

- Choose *semi-simple* element $E \equiv E^{(1)}$ and its Kernel, i.e., $\mathcal{K} = Kernel = \{x \in \hat{\mathcal{G}}/[x, E] = 0\}$,

$$E^{(1)} = E_\alpha + \lambda E_{-\alpha} = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}, \qquad \mathcal{K}_{2n+1} = \{\lambda^n (E_\alpha + \lambda E_{-\alpha})\}.$$

such that $\hat{\mathcal{G}} = \mathcal{K} \oplus \mathcal{M}$.

- Define $A_0 \in \mathcal{M} \cap \mathcal{G}_0$, i.e.,

$$A_0 = v(x,t)h^{(0)} = \partial_x \phi(x,t) h^{(0)}$$

Lax operator $L = \partial_x + E^{(1)} + A_0$,

- Define 2-D Gauge potentials

$$
\begin{aligned}
A_x &= E^{(1)} + A_0, \\
A_{t_{MN}} &= D^{(N)} + D^{(N-1)} + \cdots + D^{(0)} + \cdots D^{(-M)}
\end{aligned}
$$

$$D^{(j)} \in \mathcal{G}_j, \qquad A_0 \in \mathcal{M} \subset \mathcal{G}_0 \quad \text{Image.}$$

- Zero Curvature Equation for *Hierarchy*

$$[\partial_x + E^{(1)} + A_0, \partial_{t_{MN}} + D^{(N)} + D^{(N-1)} \cdots + D^{(0)} + \cdots D^{(-M)}] = 0,$$

Decompose and solve grade by grade, i.e.,

- $$\begin{aligned} [E^{(1)}, D^{(N)}] &= 0, \\ [E^{(1)}, D^{(N-1)}] + [A_0, D^{(N)}] + \partial_x D^{(N)} &= 0, \\ \vdots &\quad \vdots \\ [E^{(1)}, D^{(-1)}] + [A_0, D^{(0)}] + \partial_x D^{(0)} - \partial_{t_{MN}} A_0 &= 0, \\ \vdots &\quad \vdots \\ [A_0, D^{(-M)}] + \partial_x D^{(-M)} &= 0, \end{aligned}$$
- Solving recursively for $D^{(i)}$, get eqn. of motion

$$\partial_{t_{MN}} A_0 - \partial_x D^{(0)} - [A_0, D^{(0)}] - [E^{(1)}, D^{(-1)}] = 0,$$

- Positive Hierarchy: $M = 0$, i.e.,

$$\begin{aligned} [E^{(1)}, D^{(N)}] &= 0, \\ [E^{(1)}, D^{(N-1)}] + [A_0, D^{(N)}] + \partial_x D^{(N)} &= 0, \\ \vdots \quad &\quad \vdots \\ [A_0, D^{(0)}] + \partial_x D^{(0)} - \partial_{t_N} A_0 &= 0, \end{aligned}$$

- In particular,

$$D^{(N)} \in \mathcal{K}_{2n+1} \longrightarrow N = 2n+1$$

Solving recursively for $D^{(i)}$ get eqn. of motion

$$\partial_{t_N} A_0 - \partial_x D^{(0)} - [A_0, D^{(0)}] = 0,$$

$N = 3,$

$$4\partial_{t_3} v = v_{3x} - 6v^2 v_x, \qquad mKdV$$

$N = 5,$

$$16\partial_{t_5} v = v_{5x} - 10v^2 v_{3x} - 40 v v_x v_{2x} - 10 v_x^3 + 30 v^4 v_x,$$

$N = 7,$

$$\begin{aligned} 64\partial_{t_7} v &= v_{7x} - 182 v_x v_{2x}^2 - 126 v_x^2 v_{3x} - 140 v v_{2x} v_{3x} \\ &- 84 v v_x v_{4x} - 14 v^2 v_{5x} + 420 v^2 v_{3x} + 560 v^3 v_x v_{2x} \\ &+ 70 v^4 v_{3x} - 140 v^6 v_x \end{aligned}$$

$\cdots$, etc.

All Eqns. derived from the same affine structure, i.e., $\hat{\mathcal{G}}$, Q, $E^{(1)}$.

Remark

- Vacuum Solution is $v = const = 0$
- zero curvature for vacuum solution becomes

$$[\partial_x + E^{(1)}, \partial_{t_N} + E^{(N)}] = 0, \qquad [E^{(1)}, E^{(N)}] = 0$$

- and imply pure gauge potentials, i.e.,

$$A_{x,vac} = E^{(1)} = T_0^{-1}\partial_x T_0, \qquad A_{t_N,vac} = E^{(N)} = T_0^{-1}\partial_{t_N} T_0$$

- where T_0 carries explicit space-time dependence,

$$T_0 = e^{xE^{(1)}} e^{t_N E^{(N)}}.$$

- Zero Curvature Equation for *Negative Hierarchy*

$$[\partial_x + E^{(1)} + A_0, \partial_{t_{-M}} + D^{(-M)} + D^{(-M+1)} + \cdots + D^{(-1)}] = 0.$$

- Lowest grade projection,

$$\partial_x D^{(-M)} + [A_0, D^{(-M)}] = 0$$

yields a nonlocal eqn. for $D^{(-M)}$. **No condition upon M.**

- The second lowest projection of grade $-M+1$ leads to

$$\partial_x D^{(-M+1)} + [A_0, D^{(-M+1)}] + [E^{(1)}, D^{(-M)}] = 0$$

and determines $D^{(-M+1)}$.

- The same mechanism works recursively until we reach the zero grade equation

$$\partial_{t_{-M}} A_0 + [E^{(1)}, D^{(-1)}] = 0$$

which gives the *time evolution* for the field in A_0 according to time t_{-M}.

- Simplest Example $t_{-M} = t_{-1}$.

$$\partial_x D^{(-1)} + [A_0, D^{(-1)}] = 0,$$

$$\partial_{t_{-1}} A_0 - [E^{(1)}, D^{(-1)}] = 0.$$

- Solution is

$$D^{(-1)} = B^{-1} E^{(-1)} B, \qquad A_0 = B^{-1} \partial_x B, \qquad B = \exp(\mathcal{G}_0)$$

- The time evolution is then given by the Leznov-Saveliev equation,

$$\partial_{t_{-1}} \left(B^{-1} \partial_x B \right) = [E^{(1)}, B^{-1} E^{(-1)} B]$$

- For $\hat{sl}(2)$ with principal gradation $Q = 2\lambda \frac{d}{d\lambda} + \frac{1}{2}h$, yields the (relativistic) sinh-Gordon equation

$$\partial_{t_{-1}} \partial_x \phi = e^{2\phi} - e^{-2\phi}, \qquad B = e^{\phi h}.$$

where $\quad t_{-1} = z, \quad x = \bar{z}, \quad A_0 = vh = \partial_x \phi h.$

- **No restriction for Negative even Hierarchy**

Next simplest example $t_{-n} = t_{-2}$

JFG, G S França, G R de Melo and A H Zimerman, *J. of Phys.* **A42**,(2009), 445204

$$\begin{aligned} \partial_x D^{(-2)} + [A_0, D^{(-2)}] &= 0, \\ \partial_x D^{(-1)} + [A_0, D^{(-1)}] + [E^{(1)}, D^{(-2)}] &= 0, \\ \partial_{t_{-2}} A_0 - [E^{(1)}, D^{(-1)}] &= 0. \end{aligned}$$

- Propose solution of the form

$$\begin{aligned} D^{(-2)} &= c_{-2}\lambda^{-1} h, \\ D^{(-1)} &= a_{-1}\left(\lambda^{-1} E_\alpha + E_{-\alpha}\right) + b_{-1}\left(\lambda^{-1} E_\alpha - E_{-\alpha}\right). \end{aligned}$$

$t = t_{-2}$ Equation

- Get $c_{-2} = const$ and

$$\begin{aligned} a_{-1} + b_{-1} &= 2c_{-2} \exp(-2d^{-1}v) d^{-1} \left(\exp(2d^{-1}v) \right), \\ a_{-1} - b_{-1} &= -2c_{-2} \exp(2d^{-1}v) d^{-1} \left(\exp(-2d^{-1}v) \right), \end{aligned}$$

where $A_0 = vh$ and $d^{-1}v = \int^x v(x')dx'$.

- Equation of motion is (integral eqn.)

$$\partial_{t_{-2}} v = -2c_{-2} e^{-2\phi} d^{-1} \left(e^{2\phi} \right) - 2c_{-2} e^{2\phi} d^{-1} \left(e^{-2\phi} \right)$$

where $\phi \equiv d^{-1} v = \int^x v(x')dx'$.

- In the same way we find

$$\partial_{t_{-3}} v = 4e^{-2\phi} d^{-1} \left(e^{2\phi} d^{-1} (\sinh 2\phi) \right) + 4e^{2\phi} d^{-1} \left(e^{-2\phi} d^{-1} (\sinh 2\phi) \right)$$

and so on ...

- Vacuum structure for sinh-Gordon, and all other negative odd flows, $v = \phi = 0$.

- Vacuum for $t = t_{-2}$ eqn. and all other negative even flows, $v = v_0 \neq 0, \qquad \phi = v_0 x$

The **Soliton solutions** are constructed from the vacuum solution by **gauge transformation** (which preserves the zero curvature condition), i.e.,

$$A_\mu = \Theta^{-1} A_{\mu,vac} \Theta + \Theta^{-1} \partial_\mu \Theta,$$

where

$$A_\mu = \Theta^{-1}\partial_\mu \Theta, \qquad T = \Theta T_0, \qquad A_{\mu,vac} = T_0^{-1}\partial_\mu T_0$$

we may choose $\Theta = \Theta_+ = e^{\theta_0} e^{\theta_1} \cdots$ or $\Theta = \Theta_- = e^{\theta_{-1}} e^{\theta_{-2}} \cdots$, $\theta_i \in \mathcal{G}_i$.

It then follows that $T = \Theta_+ T_0 = \Theta_- T_0 g$,

$$\Theta_-^{-1}\Theta_+ = T_0^{-1} g T_0, \qquad e^{\theta_0} = Be^{\nu \hat{k}}$$

where we have extended the loop algebra to the full **central extended Kac-Moody** algebra

$$[h^{(m)}, h^{(n)}] = \hat{k} m \delta_{m+n,0}$$

$$[h^{(m)}, E_{\pm\alpha}^{(n)}] = \pm 2 E_{\pm\alpha}^{(n+m)}, \qquad [E_{\alpha}^{(m)}, E_{-\alpha}^{(n)}] = h^{(m+n)} + \hat{k} m \delta_{m+n,0},$$

in order to introduce highest weight states $|\lambda_i >, i = 0, 1$, i.e., $\mathcal{G}_> |\lambda_i >= 0$ and $< \lambda_i | \mathcal{G}_< = 0$,

$$B \to Be^{\nu \hat{k}}, \qquad A_0 \to A_0 + \partial_x \nu \hat{k}$$

such that

$$< \lambda | Be^{\nu \hat{k}} | \lambda > \;\; = \;\; < \lambda | T_0^{-1} g T_0 | \lambda > .$$

- The solution for mKdV hierarchy is then given by

$$\begin{aligned} e^{-\nu} &= <\lambda_0|T_0^{-1} g T_0|\lambda_0> \equiv \tau_0, \\ e^{-\phi-\nu} &= <\lambda_1|T_0^{-1} g T_0|\lambda_1> \equiv \tau_1 \end{aligned}$$

and hence,

$$v = -\partial_x \ln\left(\frac{\tau_0}{\tau_1}\right), \qquad v = \partial_x \phi.$$

where

$$T_0 = e^{xA_{x,vac}} e^{t_M A_{t_M,vac}}, \qquad g = e^{F(\gamma)},$$

and $F(\gamma)$ is an eigenvector (vertex operator) of $b_M = A_{t_M,vac}$ and $b_1 = A_{x,vac}$, i.e.,

$$[b_M, F(\gamma)] = w_M(\gamma) F(\gamma) .$$

- We find that the *one-soliton solution for zero vacuum* , $v_0 = 0$, of the form,

$$\tau_0 = 1 + C_0 \rho(\gamma, v_0), \qquad \tau_1 = 1 + C_1 \rho(\gamma, v_0)$$

solves all eqns. within the mKdV hierarchy for

$$A_x^{vac} = b_1 = E^{(1)}, \qquad A_{t_N}^{vac} = b_N = E^{(N)},$$

and $w_1 = 2\gamma, \quad w_N = 2\gamma^N$, i.e.,

$$v = -\partial_x \ln\left(\frac{1 + C_1\rho}{1 + C_0\rho}\right).$$

where

$$\rho(\gamma, v_0) = \exp\left\{\gamma x + \gamma^N t_N\right\}.$$

- The same works for *multi-soliton* solutions, ie., $g = \Pi e^{F_i(\gamma_i)}$.

- Non Zero Vacuum $v_0 \neq 0$,

$$\begin{aligned}
A_x^{vac} &= E^{(1)} + v_0 h^{(0)}, \\
A_{t_3}^{vac} &= E^{(3)} + v_0 h^{(1)} - \frac{1}{2} v_0^2 (E^{(1)} + v_0 h^{(0)}), \\
A_{t_5}^{vac} &= E^{(5)} + v_0 h^{(2)} - \frac{1}{2} v_0^2 (E^{(3)} + v_0 h^{(1)}) + \frac{3}{5} v_0^4 (E^{(1)} + v_0 h^{(0)}), \\
\vdots \quad & \quad \vdots \\
A_{t_{2n+1}}^{vac} &= \sum_{j=1}^{n} c_j v_0^{2n-2j} B^{2j+1},
\end{aligned}$$

where $B^{2j+1} = E^{(2j+1)} + v_0 h^{(j)} \rightarrow$ abelian algebra, i.e.,

$$[\ B^{(M)},\ \ B^{(N)}\ ] = 0.$$

- Notice that B^{2j+1} contains terms of different $Q-$ grade.

- One soliton Solution for $v_0 \neq 0$,
- Deformed Vertex Operator

$$[E^{(2n+1)} + v_0 h^{(n)}, F(\gamma, v_0)] = w_N(\gamma, v_0) F(\gamma, v_0),$$

$$v(x, t_N) - v_0 = \partial_x ln \left(\frac{1+ < \mu_1 | F(\gamma, v_0) | \mu_1 > \rho_N(x, t_N, v_0)}{1+ < \mu_0 | F(\gamma, v_0) | \mu_0 > \rho_N(x, t_N, v_0)} \right),$$

where

$$\begin{aligned} \rho_3 &= e^{2\gamma x + (2\gamma^3 - 3v_0\gamma)t_3}, \\ \rho_5 &= e^{2\gamma x + (2\gamma^5 - 5v_0^2\gamma^3 + \frac{15}{4}v_0^4\gamma)t_5}, \\ \vdots & \quad \vdots \end{aligned}$$

- One soliton Solution for $v_0 \neq 0, \quad t = t_{-2n}$,

$$v(x, t_N) - v_0 = \partial_x ln\left(\frac{\tau_0}{\tau_1}\right),$$

where

$$\tau_0 = 1 + \frac{v_0 + \gamma}{2\gamma}\rho, \qquad \tau_0 = 1 + \frac{v_0 - \gamma}{2\gamma}\rho$$

anf ρ is given by

$$\rho(x, t_{-2n}, v_0) = e^{2\gamma x + \frac{2\gamma}{v_0(\gamma^2 - v_0^2)^n} t_{-2n}}, \qquad v_0 \neq 0$$

- Key ingredient $\longrightarrow$ **Lie Algebra** $\mathcal{G}$:

$$[h, E_{\pm\alpha}] = \pm 2E_{\pm\alpha}, \qquad [E_\alpha, E_{-\alpha}] = h,$$

- Introduce spectral parameter $\lambda \longrightarrow$ **Loop algebra** $\tilde{\mathcal{G}}$:

$$[h^{(m)}, E_{\pm\alpha}^{(n)}] = \pm 2E_{\pm\alpha}^{(n+m)}, \qquad [E_\alpha^{(m)}, E_{-\alpha}^{(n)}] = h^{(m+n)}$$

- soliton solution requires highest weight states $\longrightarrow$ full **central extended Kac-Moody algebra** $\hat{\mathcal{G}}$:

$$[h^{(m)}, h^{(n)}] = \hat{k} m \delta_{m+n,0}$$

$$[h^{(m)}, E_{\pm\alpha}^{(n)}] = \pm 2E_{\pm\alpha}^{(n+m)}, \qquad [E_\alpha^{(m)}, E_{-\alpha}^{(n)}] = h^{(m+n)} + \hat{k} m \delta_{m+n,0},$$

Consider the general Affine Kac-Moody algebra

$$[T_a^{(m)}, T_b^{(n)}] = f_{abc} T_c^{(n+m)} + \hat{k} m \delta_{m+n,0} \delta_{ab},$$

obtained as(Laurent) Fourier coefficients of currents

$$J_a(x) = J_a(x+L) = \frac{2\pi\hbar}{L} \sum T_a^{(n)} e^{-\frac{2\pi i x}{L} n},$$

Using $\delta(x-y) = \frac{2\pi}{L} \sum e^{\frac{2\pi i (x-y)}{L} n}$ we find the current algebra,

$$[J_a(x), J_b(y)] = \hbar f_{abc} J_c(x) \delta(x-y) + \hbar^2 \hat{k} \delta'(x-y) \delta_{ab},$$

explicitating the *quantum mechanical character* of the central term, i.e., in classical limit $\hbar \to 0$ and $\frac{1}{i\hbar}[,] \longrightarrow \{,\}_{PB}$

Introduce quadratic operators (e.g. *Energy Momentum Tensor*)

$$L(z) = \frac{1}{2k} \sum_{a=1}^{dim\ \mathcal{G}} T_a(z) T_a(z) = \sum_{n=-\infty}^{+\infty} L_n z^{-n}, \quad z = e^{\frac{2\pi i x}{L}}$$

where $T_a(z) = T_a^{(n)} z^{-n}$, we find $\quad [L^{(m)}, T_b^{(n)}] = -nT_b^{(n+m)}$
and

$$[L^{(m)}, L^{(n)}] = (m-n)L^{(n+m)}$$

Is known as the centerless *Virasoro Algebra* (or Witt algebra).

In order to introduce central terms for Virasoro algebra need to introduce *Normal Ordering*,

$$
\begin{aligned}
: T_a^{(-n)} T_a^{(n)} : \; &= \; T_a^{(-n)} T_a^{(n)}, n > 0, \\
&= \; T_a^{(n)} T_a^{(-n)}, n < 0
\end{aligned}
$$

and

$$L(z) = \frac{1}{2k + Q_\psi} \sum_{a=1}^{dim \;\; \mathcal{G}} : T_a(z) T_a(z) := \sum_{n=-\infty}^{+\infty} L_n z^{-n},$$

where (Virasoro Algebra)

$$[L^{(m)}, L^{(n)}] = (m-n) L^{(n+m)} + \frac{c}{12} m(m^2 - 1)\delta_{m+n,0}$$

$Q_\psi \delta_{ab} = \sum_{i,j} f_{aij} f_{bij}, \qquad c = \frac{2K dim\mathcal{G}}{2K+Q_\psi}.$

1. Construction of Integrable Hierarchy from 3 basic Lie Algebraic ingredients, $\mathcal{G}$, Q and E.

2. Construction of Soliton Solutions in terms of representation of Kac-Moody algebras.

3. Extend algebra to incorporate Virasoro Algebra, which generates conformal transformations

4. Aplications to conformally invariant models, e.g., string theory, critical exponents in 2-D statistical systems, etc