Infinite Dimensional Algebras, Conformal Invariance and Applications - Lecture 3

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In general Infinite Dimensional Algebras are relevant to

- Construction of 2-D non-linear *time evolution integrable* equations (Integrable Hierarchies) associated to *graded Affine Lie algebraic* structure.

- Representation Theory of Infinite Dimensional Algebras and the Systematic Construction of Soliton Solutions.

- Conformal Transformation in 2D and the Virasoro Algebra. String Theory, Critical Exponents in 2D Statistical Models, etc

- Consider 2-D plane

$$(x, y) \;\; plane \;\; coordinates \rightarrow (z, \bar{z}) \;\; complex \;\; plane,$$

- infinitesimal conformal transformation

$$\begin{aligned} z &\rightarrow F(z) = z + \sum_{n \in Z} \epsilon_n z^{n+1} \\ \bar{z} &\rightarrow \bar{F}(\bar{z}) = \bar{z} + \sum_{n \in Z} \bar{\epsilon}_n \bar{z}^{n+1} \end{aligned} \tag{1}$$

where $F, \bar{F}$ are analytic functions, e.g.,

- $\delta_0 z = \epsilon_0 z$, dilatation
- $\delta_{-1} z = \epsilon_{-1}$, translation, $\cdots$ etc

Spin Systems in 2-D lattice, Ising, Potts, $\cdots$ etc

- At the **critical temperature system becomes independent of lattice spacing** and hence **conformal invariant.**
- Critical Exponents $\longrightarrow$ Representations of Conformal Algebra

- Relativistic Point Particle $\rightarrow$ one parameter position function,

$$X_\mu = X_\mu(t).$$

Action $\rightarrow$ length of path followed by point particle, i.e.,

$$S \sim \int \sqrt{\frac{\partial X_\mu}{\partial t}\frac{\partial X^\mu}{\partial t}} dt = \int \sqrt{1-(v/c)^2}$$

- Relativistic String $\rightarrow$ two parameter position function,

$$X_\mu = X_\mu(\sigma, \tau)$$

Action (Nambu) $\rightarrow$ area spanned by the string, i.e.,

$$S \sim \int \sqrt{\left(\frac{\partial X_\mu}{\partial \sigma}\right)^2 \left(\frac{\partial X^\nu}{\partial \tau}\right)^2 - \left(\frac{\partial X_\mu}{\partial \sigma}\frac{\partial X^\mu}{\partial \tau}\right)^2} d\sigma d\tau$$

Nambu Action is reparametrization (conformal) invariant,

$$\sigma \rightarrow \tilde{\sigma}(\sigma, \tau), \;\; \tilde{\tau} \rightarrow \tilde{\tau}(\sigma, \tau).$$

- Consider the conformal transformation,

$$z \longrightarrow F(z), \quad F \text{ is analytic}$$

Infinitesimally,

$$z \longrightarrow z + \epsilon(z) \;\; = \;\; z + \sum_{n \in Z} \epsilon_n z^{n+1}$$

A field (“classical") transforms as

$$\phi(z) \to \phi\left(z + \epsilon(z)\right) \;\; = \;\; \phi(z) + \;\; \epsilon_n z^{n+1} \frac{\partial \phi}{\partial z}$$

$$\delta_n \phi = [L_n, \phi] \;\; = \;\; z^{n+1} \frac{\partial \phi}{\partial z}$$

where $L_n = -z^{n+1} \frac{d}{dz}$ are the generators of the conformal algebra.

- E.g.,

$$
\begin{aligned}
L_0 &= z\frac{d}{dz} \longrightarrow \delta_0 z = \epsilon_0 z, \quad \textit{dilatation}, \\
L_{-1} &= \frac{d}{dz} \longrightarrow \delta_{-1} z = \epsilon_{-1}, \quad \textit{translation}, \\
\cdots \textit{etc}
\end{aligned}
$$

- Algebra of Conformal Generators $\longrightarrow$ Centerless Virasoro Algebra (Witt Algebra)

$$[L_m, L_n] = (m-n)L_{m+n}$$

- **Question:**. What is the most general Conformal Algebra satisfying Jacobi Identities???

- The Virasoro Algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0}$$

where $[c, L_m] = 0$ and c characterizes the central term.

- Primary Fields

$$\phi(z,\bar{z}) \to \left(\frac{\partial F}{\partial z}\right)^{\Delta} \left(\frac{\partial \bar{F}}{\partial \bar{z}}\right)^{\bar{\Delta}} \phi\left(F(z), \bar{F}(\bar{z})\right)$$

Δ and $\bar{\Delta}$ are anomalous dimensions.

Consider only z for simplicity,

- Infinitesimally $\longrightarrow$ generalization of "classical ' field transformation.

$$\phi(z) \to (1 + \epsilon_n \Delta (n+1) z^n) \left(\phi(z) + \epsilon_n z^{n+1} \frac{\partial \phi}{\partial z} \right),$$

or

$$\delta_n \phi \equiv [L_n, \phi] = z^n \left(z \frac{\partial \phi}{\partial z} + \Delta (n+1) \phi \right)$$

- Consider a “quantum"field

$$\phi(z) = \sum_{n \in Z} \varphi_n z^{-n}$$

acting in a Fock space generated by vacuum state $|0>$,

$$\varphi_n|0>= 0, \quad n > 0.$$

- Field transformation yields

$$[L_n, \varphi_k] = (\Delta(n+1) - (n+k))\, \varphi_{n+k}$$

Since $\varphi_k|0>=0, \quad k=1,2,\cdots$ it follows that

- $n=-1$ and $k=1$,

$$[L_{-1},\varphi_1]|0>=0 \quad \to \quad L_{-1}|0>=0$$

- $n=0$ and $k=1$,

$$[L_0,\varphi_1]|0>=(\Delta-1)\varphi_1|0>=0 \quad \to \quad L_0|0>=0$$

- $n=1$ and $k=1$,

$$[L_1,\varphi_1]|0>=2(\Delta-1)\varphi_2|0>=0 \quad \to \quad L_1|0>=0$$

- **In general**

$$L_n|0>=0, \quad n\geq -1$$

Highest Weight States

Consider now the state, $\varphi_0|0>$

- $$\begin{aligned} L_0\,\varphi_0|0> &= ([L_0,\varphi_0]-\varphi_0 L_0)\,|0>=\Delta\varphi_0|0>, \\ L_n\,\varphi_0|0> &= ([L_n,\varphi_0]-\varphi_0 L_n)\,|0>=0, \quad n>0 \end{aligned}$$

- each *primary field* φ_0 generates a *highest weight state* $|\Delta>$, i.e.,

$$|\Delta>\equiv\varphi_0|0>.$$

- Representations

$$L_0|\Delta >= \Delta|\Delta >, \qquad L_n|\Delta >= 0, \;\; n > 0$$

- Excited states are

$$(L_{-n_1})^{m_1} \, (L_{-n_2})^{m_2} \, \cdots (L_{-n_k})^{m_k} \, |\Delta > \equiv |\Delta - N >$$

are eigenstates of L_0 with eigenvalue $\Delta - N$,

$$N = n_1 m_1 + n_2 m_2 + \cdots n_k m_k$$

- Two point function

$$< 0|\phi(z_1)\phi(z_2)|0 > = \; f(z_1, z_2)$$

It follows from $[L_{-1}, \phi(z_i)] = \frac{\partial}{\partial z_i}\phi(z_i)$,

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$$\begin{aligned} 0 &= \; < 0|L_{-1}\phi(z_1)\phi(z_2)|0 > \\ &= \; < 0|[L_{-1}, \phi(z_1)]\phi(z_2)|0 > + < 0|\phi(z_1)[L_{-1}, \phi(z_2)]|0 > \\ &= \; (\frac{\partial}{\partial z_1} - \frac{\partial}{\partial z_2})f(z_1, z_2) \end{aligned}$$

Henceforth $f(z_1, z_2) = f(z_1 - z_2)$.

Correlation Functions and Critical Exponents

- It follows from $[L_0, \phi(z_i)] = z_i \frac{\partial}{\partial z_i}\phi(z_i) + \Delta\phi(z_i)$ and

$$
\begin{aligned}
0 &= \ < 0|L_0\phi(z_1)\phi(z_2)|0 > \\
&= \ < 0|[L_0, \phi(z_1)]\phi(z_2)|0 > + < 0|\phi(z_1)[L_0, \phi(z_2)]|0 >
\end{aligned}
$$

$$
0 = (z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}) f(z_1 - z_2) + 2\Delta f(z_1 - z_2)
$$

- Integrating we find

$$
f(z_1 - z_2) = \frac{1}{(z_1 - z_2)^{2\Delta}}
$$

Considering both coordinates $(z, \bar{z})$ we find

- $$< 0|\phi(z_1, \bar{z}_1)\phi(z_2, \bar{z}_2)|0 > = \; r^{-2(\Delta+\bar{\Delta})} e^{-2i\theta(\Delta-\bar{\Delta})},$$

  where

  $$z_1 - z_2 = r^{i\theta}, \qquad \bar{z}_1 - \bar{z}_2 = r^{-i\theta},$$

- $$\Delta + \bar{\Delta} = \text{critical exponent}, \qquad \Delta - \bar{\Delta} = \text{spin}$$

- Unitarity $\longrightarrow$ all states of the form

$$|\psi>= (L_{-n_1})^{m_1}\,(L_{-n_2})^{m_2}\,\cdots(L_{-n_k})^{m_k}\,|\Delta>$$

have positive norm, i.e.,

$$<\psi|\psi>=|\psi|^2\geq 0$$

- Simplest state $|\psi>=L_{-n}|\Delta>$. Unitarity requires

$$<\Delta|L_nL_{-n}|\Delta>=<\Delta|[L_n,L_{-n}]|\Delta>=2\Delta+\frac{c}{12}n(n^2-1)\geq 0$$

In particular implies (see diagram 1)

$$\begin{aligned} n=1 &\quad\longrightarrow\quad \Delta\geq 0 \\ n>>1 &\quad\longrightarrow\quad c\geq 0 \end{aligned}$$

- Consider a family of states $|\psi >= L_{-1}^2|\Delta > +bL_{-2}|\Delta >$ parametrized by a complex coefficient b.

$$\begin{aligned} < \psi|\psi > &= < \Delta|L_1^2 L_{-1}^2|\Delta > +b < \Delta|L_1^2 L_{-2}|\Delta > \\ &+ b^* < \Delta|L_2 L_{-1}^2|\Delta > +bb^* < \Delta|L_2 L_{-2}|\Delta >, \end{aligned}$$

Minimizing with respect to b and b^* we find

$$b = b^* = -\frac{< \Delta|L_1^2 L_{-2}|\Delta >}{< \Delta|L_2 L_{-2}|\Delta >} = \frac{6\Delta}{\Delta + c/2}$$

Unitarity condition implies

$$< \psi|\psi >_{min} = 4\Delta\,(8\Delta^2 + \Delta(c-5) + c/2) \geq 0$$

(see diagram 2)

Unitary Representations

- Observe that minimizing $< \psi|\psi >$ with respect to b and b^* is equivalent to

$$det \begin{bmatrix} < \Delta|L_1^2 L_{-1}^2|\Delta > & < \Delta|L_1^2 L_{-2}^2|\Delta > \\ < \Delta|L_2 L_{-1}^2|\Delta > & < \Delta|L_2 L_{-2}|\Delta > \end{bmatrix}$$

- More and more states are introduced $\longrightarrow$ forbidden region (for $0 \leq c \leq 1$) increases (diagrams 3 and 4).

- Only intersection points are allowed

- General Solution
- $c \geq 1, \qquad \Delta > 0$
- Discrete series, $0 \leq c < 1$,

  Unitarity $\longrightarrow$ *Kac-Determinant* [1]

[1] Friedan, Qiu and Shenker, Phys. Rev. Lett. 52, (1984),1575

- Discrete series, $0 \leq c < 1$,
  - $$c = 0 \qquad \Delta = 0$$
  - $$c = 1/2, \qquad \Delta = 0, \quad \frac{1}{16}, \quad \text{or} \quad \frac{1}{2}$$
  - $$c = 7/10, \qquad \Delta = 0, \frac{3}{80}, \frac{1}{10}, \frac{7}{16}, \frac{3}{5}, \frac{3}{2}$$

- In Compact form $\longrightarrow$ Kac Determinant,

$$c = 1 - \frac{6}{(m+2)(m+3)}, \qquad m = 0, 1 \cdots$$

and

$$\Delta = \Delta_{p,q} = \frac{((m+3)p - (m+2)q)^2 - 1}{4(m+2)(m+3)}$$

$p = 1, 2, \cdots, m+1 \qquad , q = 1, 2, \cdots p.$

- Consider the two Virasoro algebras with $\mathbf{c} = \mathbf{1/2}$

$$\begin{aligned} \Delta = \bar{\Delta} = 1/16, \qquad & x = \Delta + \bar{\Delta} = 1/8 \\ \Delta = \bar{\Delta} = 1/2, \qquad & x = \Delta + \bar{\Delta} = 1 \\ \Delta = 1/2, \bar{\Delta} = 0, \qquad & x = \Delta + \bar{\Delta} = 1/2 \end{aligned}$$

  $x = 1/8,\ 1$ and $\ 1/2$ correspond to the Critical Exponents for the **Ising Model.**
- For $\mathbf{c} = \mathbf{7/10}$ the critical exponents are

$$\begin{aligned} \Delta = \bar{\Delta} = 3/80, \qquad & x = \Delta + \bar{\Delta} = 3/40 \\ \Delta = \bar{\Delta} = 1/10, \qquad & x = \Delta + \bar{\Delta} = 1/5 \\ \Delta = \bar{\Delta} = 7/16, \qquad & x = \Delta + \bar{\Delta} = 7/8 \\ \Delta = \bar{\Delta} = 3/5, \qquad & x = \Delta + \bar{\Delta} = 6/5 \\ \Delta = \bar{\Delta} = 3/2, \qquad & x = \Delta + \bar{\Delta} = 3 \end{aligned}$$

  correspond to the **Tricritical Ising Model**, etc

- $c = 0$, "Classical" theory $\longrightarrow \Delta = 0$, "Classical" Fields $\phi(z)$.
- Highest weight states, $|\Delta > \longrightarrow \varphi_0|0 >$, Primary fields
- Discrete series for $0 \leq c < 1$ (acumulates in 1) leads to critical exponents.
- $c > 1$, String theories (e.g., c=10,26).