Infinite Dimensional Algebras Applications to Conformal Invariance and Integrable Models - Lecture 1

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Plan of the Course

- **Lecture 1**- Finite Dimensional Lie Algebras. Symmetries in physics, e.g. rotations, etc. Prototype is Angular momentum
- **Lecture 2**- Construction of 2-D non-linear time evolution integrable equations (Integrable Hierarchies) associated to *graded Affine Lie algebraic* structure, e.g., sinh-Gordon, mKdV eqns., etc .
- Representation Theory of Infinite Dimensional Algebras and the Systematic Construction of Soliton Solutions.
- **Lecture 3**- Virasoro Algebra and Conformal Transformation in 2D. String Theory, Critical Exponents in 2D Statistical Models, etc

- Symmetries in Physics are in general described by a Lie algebraic structure, e.g.,

- Translations $X'_\mu = \quad X_\mu + a_\mu$

$$X'_\mu = \quad X_\mu + a_\nu p^\nu(X_\mu), \qquad p^\mu = \partial_\mu$$

- Dilatations, $X'_\mu = \quad \lambda X_\mu$

$$X'_\mu = \lambda \mathcal{D}(X_\mu), \qquad \mathcal{D} = x^\mu \frac{d}{dx^\mu}$$

- Rotations (Boosts), $X'_\mu = \quad X_\mu + \epsilon_{\rho\sigma} X^\sigma$,

$$X'_\mu = \quad X_\mu + \frac{i}{2}\epsilon_{\rho,\sigma}(L_{\rho,\sigma})X_\mu, \qquad L_{\rho,\sigma} = i(X_\rho\partial_\sigma - X_\sigma\partial_\rho)$$

- Spatial rotations, $x'^2 + y'^2 = x^2 + y^2$,

$$x' = x\ cos\theta + y\ sin\theta, \qquad y' = -x\ sin\theta + y\ cos\theta,$$

- Infinitesimally $\theta = \epsilon + O(\epsilon^2)$,

$$x' - x = y\ \epsilon = L\ x, \qquad y' - y = -x\ \epsilon\ = Ly,$$

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$$L \sim -x\frac{\partial}{\partial_y} + y\frac{\partial}{\partial_x},$$

- Boosts $t'^2 - x'^2 = t^2 - x^2$,

$$t' = t\ cosh\eta - x\ sinh\eta = \frac{t}{\sqrt{1-(v/c)^2}} - \frac{x\ v/c}{\sqrt{1-(v/c)^2}},$$

$$x' = -t\ sinh\eta + x\ cosh\eta = -\frac{t\ v/c}{\sqrt{1-(v/c)^2}} + \frac{x}{\sqrt{1-(v/c)^2}}.$$

- $\eta \to v/c$.

$$cosh\eta = \frac{1}{\sqrt{1-(v/c)^2}}$$

- Angular Momentum Prototype

$$\vec{J} = \vec{r} \times \vec{p} = \vec{r} \times (-i\hbar\vec{\nabla})$$

$$J_1 = y\,p_z - z\,p_y = -i\hbar\,(y\partial_z - z\partial_y)\,,$$
$$J_2 = z\,p_x - x\,p_z = -i\hbar\,(z\partial_x - x\partial_z)\,,$$
$$J_3 = x\,p_y - y\,p_x = -i\hbar\,(x\partial_y - y\partial_x)\,,$$

$J_a^\dagger = J_a$, for *classical or quantum* angular momentum.

- Angular Momentum Algebra $\to$ *Poisson Brackets or Commutators*, i.e.,

$$\{x_a\,,\,p_b\}_{PB} = \delta_{a,b} \qquad \text{or} \qquad [x_a\,,\,-i\hbar\frac{\partial}{\partial_{x_b}}] = i\hbar\,\delta_{ab}$$

$$\{J_a\,,\,J_b\}_{PB} = \epsilon_{abc}J_c \quad \longrightarrow \quad [J_a\,,\,J_b] = i\hbar\epsilon_{abc}J_c$$

in classical limit $\frac{1}{i\hbar}[\;] \to \{\;\}_{PB}$

- Quantum mechanically, angular momentum is described by 3 generators satisfying $\hbar = 1$,

$$[J_a\ ,\ J_b] = i\epsilon_{abc}J_c, \qquad \epsilon_{123} = 1, \qquad J_a^\dagger = J_a$$

Define

$$J_\pm = J_1 \pm iJ_2, \qquad J_\pm^\dagger = J_\mp$$

such that

$$[J_3,\ J_\pm] = \pm\, J_\pm$$

Observables in QM are described by diagonal operators, i.e.,

$$J_3\,|m\rangle = m\,|m\rangle$$

- Consider now the quantity $J_{\pm}\left|m\right\rangle$,

$$J_3(J_{\pm}\left|m\right\rangle) = ([J_3,\ J_{\pm}] + J_{\pm}J_3)\left|m\right\rangle = (m \pm 1)J_{\pm}\left|m\right\rangle ,$$

- $J_{\pm}\left|m\right\rangle \rightarrow$ eigenstates of J_3 with eigenvalues $m \pm 1$.

- $J_{\pm} \rightarrow$ creates and anahillates **one quanta** of angular momentum respectively.

- Seek for Matrices satisfying algebra $\longrightarrow$ Representations.
- Physics require finite dimensional representation, i.e., require *highest weight state* $|j\rangle$ (vacuum) such that,

$$J_+ |j\rangle = 0$$

excited states are constructed by

$$(J_-)^k |j\rangle \sim |j-k\rangle$$

by symmetry, $J_- |-j\rangle = 0$. (space is *compact*, c.f. harm oscilator)
- Representation consists of $2j+1$ states $|j,m\rangle$,

$$\{|j,-j\rangle\,,\ |j,-j+1\rangle\,,\ \cdots |j,j-1\rangle\,,\ |j,j\rangle\}$$

- It therefore follows that $j \in \mathbb{Z}$ or $j \in \mathbb{Z} + \frac{1}{2}$.

- Casimir $J^2 = J_1^2 + J_2^2 + J_3^2 = J_3^2 + \frac{1}{2}(J_- J_+ + J_+ J_-)$ such that
$$[J^2,\ J_a] = 0, \quad \mathrm{a} = 1, 2, 3.$$
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$$J^2 \left| j, m \right\rangle = j(j+1) \left| j, m \right\rangle$$
- Diagram.

- Define matrices

$$(J_3)_{n,m} = \langle j, m| \, J_3 \, |j, n\rangle = m \, \delta_{m,n}$$

$$(J_+)_{n,m} = \langle j, m| \, J_+ \, |j, n\rangle = N^{(+)}_{n,m} \;\; \delta_{m,n+1}$$

$$(J_-)_{n,m} = \langle j, m| \, J_- \, |j, n\rangle = N^{(-)}_{n,m} \;\; \delta_{m,n-1}$$

- Normalization

$$N^{(+)}_{n,m} = \sqrt{(j-m)(j+m+1)}, \qquad N^{(-)}_{n,m} = \sqrt{(j+m)(j-m+1)}$$

Examples

- $j = 1/2$

$$J_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

- $j = 1$

$$J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad J_+ = \sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad J_- = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

- etc

General Structure of Lie Algebras

- Observables $\{h_1, h_2, \cdots, h_r, \quad r = rank\}$, e.g., $\quad J_3 = h_1$

$$[h_i,\ h_j] = 0, \qquad i, j = 1, \cdots, r$$

- Step operators, e.g., $\quad J_\pm = E_{\pm\alpha}, \quad$ such that

$$[h_i,\ E_{\pm\alpha}] = \pm\alpha^i E_{\pm\alpha}, \qquad \vec{\alpha} = (\alpha^1, \alpha^2, \cdots, \alpha^r)$$

$\vec{\alpha}$ is a r-dim. vector called root, (**quanta**) and

$$[E_\alpha,\ E_\beta] = \begin{cases} \epsilon(\alpha,\beta)E_{\alpha+\beta}, & \alpha+\beta = \text{ root} \\ 2\frac{\sum \alpha^i h_i}{\alpha^2}, & \alpha+\beta = 0 \\ 0, \text{ otherwise} & \end{cases}$$

Simple Examples,

- **One Observable**, $r = 1$, $\mathcal{G} = su(2)$ Angular Momentum

$$[h,\ E_{\pm\alpha}] = \pm\alpha E_{\pm\alpha}, \qquad [E_\alpha,\ E_{-\alpha}] = 2h,$$

- Root system $\rightarrow$ one dimensional vector, $\alpha = 1$, i.e.,

2 Observables, rank $r = 2$ Algebras

- $so(4) \sim su(2) \otimes su(2)$. Lorentz Algebra

$$[h_1,\ E_{\pm\alpha}] = (\pm 1)E_{\pm\alpha},$$

$$[h_2,\ E_{\pm\beta}] = (\pm 1)E_{\pm\beta},$$

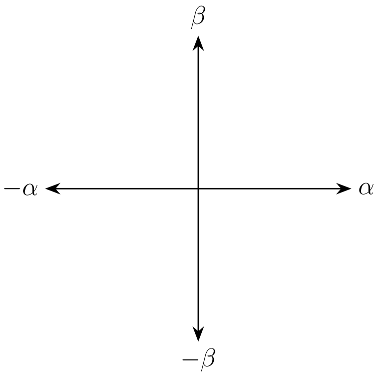
$$[h_1,\ E_{\pm\beta}] = 0,$$

$$[h_2,\ E_{\pm\alpha}] = 0.$$

$$[E_\alpha,\ E_{-\alpha}] = 2h_1,$$

$$[E_\beta,\ E_{-\beta}] = 2h_2,$$

$$[E_\alpha,\ E_\beta] = 0,$$



Representations of Lorentz algebra $|j, m\rangle \otimes |j', m' >$. Simplest Cases

- $j = j' = 0$,

$$|0, 0\rangle \otimes \left|0', 0\right\rangle = \phi, \quad \textit{scalar} \;\; (\textit{singlet})$$

- $j = 0, j' = 1/2$, denote $|\uparrow\rangle = |1/2, 1/2>, \quad |\downarrow\rangle = |1/2, -1/2>$,

$$\psi_L = \begin{pmatrix} |0,0\rangle \otimes |\uparrow\rangle \\ |0,0\rangle \otimes |\downarrow\rangle \end{pmatrix} = \begin{pmatrix} \psi_L^{(1)} \\ \psi_L^{(2)} \end{pmatrix},$$

- $j = 1/2, j' = 0$

$$\psi_R = \begin{pmatrix} |\uparrow\rangle \otimes |0,0\rangle \\ |\downarrow\rangle \otimes |0,0\rangle \end{pmatrix} = \begin{pmatrix} \psi_R^{(1)} \\ \psi_R^{(2)} \end{pmatrix}$$

ψ_R and ψ_L are Weyl Spinors (doublets).

- Dirac Spinor

$$\psi_D = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

- $j = j' = 1/2 \quad , \quad 2 \otimes 2 = 3 \oplus 1$

$$|\uparrow\rangle \otimes |\uparrow\rangle = |1, 1\rangle$$

$$\frac{1}{\sqrt{2}}(|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle) = |1, 0\rangle$$

$$|\downarrow\rangle \otimes |\downarrow\rangle = |1, -1\rangle$$

and

$$\frac{1}{\sqrt{2}}(|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle) = |0, 0\rangle$$

- Vector field A_μ

- $j = 1/2, \quad j' = 1 \quad$ and $\quad j = 1, \quad j' = 1/2$

  or $\quad su(2) \otimes [su(2) \otimes su(2)]$ and $[su(2) \otimes su(2)] \otimes su(2)$

  get spin 3/2 fields, i.e., $\quad (\psi_L)_\mu$ and $(\psi_R)_\mu$, $\rightarrow$ gravitino

$$|\uparrow\rangle \otimes |\uparrow\rangle \otimes |\uparrow\rangle = |3/2, 3/2\rangle$$

$$\frac{1}{\sqrt{3}}(|\uparrow\rangle \otimes |\downarrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\downarrow\rangle \otimes |\uparrow\rangle) = |3/2, -1/2\rangle$$

$$\frac{1}{\sqrt{3}}(|\uparrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle + |\uparrow\rangle \otimes |\downarrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\uparrow\rangle) = |3/2, 1/2\rangle$$

$$|\downarrow\rangle \otimes |\downarrow\rangle \otimes |\downarrow\rangle = |3/2, -3/2\rangle$$

- together with

$$-\frac{1}{\sqrt{6}}\left(2\left|\downarrow\right\rangle\otimes\left|\downarrow\right\rangle\otimes\left|\uparrow\right\rangle-\left|\uparrow\right\rangle\otimes\left|\downarrow\right\rangle\otimes\left|\downarrow\right\rangle-\left|\uparrow\right\rangle\otimes\left|\downarrow\right\rangle\otimes\left|\uparrow\right\rangle\right)=\left|1/2,-1/2\right\rangle$$

$$\frac{1}{\sqrt{6}}\left(2\left|\uparrow\right\rangle\otimes\left|\uparrow\right\rangle\otimes\left|\downarrow\right\rangle-\left|\uparrow\right\rangle\otimes\left|\downarrow\right\rangle\otimes\left|\uparrow\right\rangle-\left|\downarrow\right\rangle\otimes\left|\uparrow\right\rangle\otimes\left|\uparrow\right\rangle\right)=\left|1/2,1/2\right\rangle$$

and

$$-\frac{1}{\sqrt{2}}\left(\left|\uparrow\right\rangle\otimes\left|\downarrow\right\rangle\otimes\left|\downarrow\right\rangle-\left|\downarrow\right\rangle\otimes\left|\uparrow\right\rangle\otimes\left|\downarrow\right\rangle\right)=\left|1/2,-1/2\right\rangle$$

$$\frac{1}{\sqrt{2}}\left(\left|\uparrow\right\rangle\otimes\left|\downarrow\right\rangle\otimes\left|\uparrow\right\rangle-\left|\downarrow\right\rangle\otimes\left|\uparrow\right\rangle\otimes\left|\uparrow\right\rangle\right)=\left|1/2,1/2\right\rangle$$

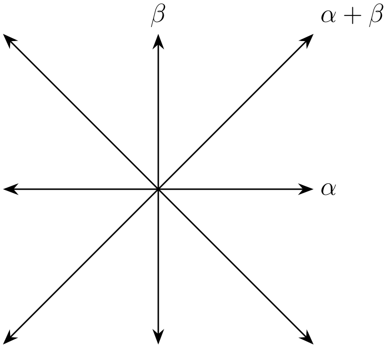
such that $\quad 2\otimes 2\otimes 2 = 4\oplus 2\oplus 2$

- $j=1, \quad j'=1$ get spin 2 field $h_{\mu,\nu}$, $\rightarrow$ graviton

- and so on

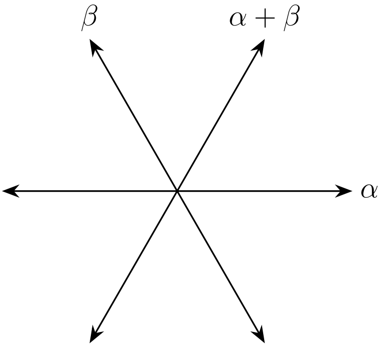
- generators of $SO(5)$

$$\{h_1, h_2, E_{\pm\alpha}, E_{\pm\beta}, E_{\pm(\alpha+\beta)}, E_{\pm(\alpha+2\beta)}\}$$



- generators of $su(3)$

$$\{h_1, h_2, E_{\pm\alpha}, E_{\pm\beta}, E_{\pm(\alpha+\beta)}\}$$



- In this case $[E_\alpha,\ E_\beta] \sim E_{\alpha+\beta}$.

- Observables, h_1, h_2 are diagonal,

$$h_1 |\lambda\rangle = \lambda^1 |\lambda\rangle , \qquad h_2 |\lambda\rangle = \lambda^2 |\lambda\rangle , \qquad \vec{\lambda} = (\lambda^1, \lambda^2)$$

- $\vec{\lambda}$ are called weights.

- Two fundamental representations 3 and $\bar{3}$ respectively,

$$\begin{array}{cc} |\mu_1\rangle , & |\mu_2\rangle , \\ E_{-\alpha_1} |\mu_1\rangle = |\mu_1 - \alpha_1\rangle , & E_{-\alpha_2} |\mu_2\rangle = |\mu_2 - \alpha_2\rangle , \\ E_{-\alpha_1-\alpha_2} |\mu_1\rangle = |\mu_1 - \alpha_1 - \alpha_2\rangle , & E_{-\alpha_1-\alpha_2} |\mu_2\rangle = |\mu_2 - \alpha_1 - \alpha_2\rangle , \end{array}$$

- Two fundamental representations with weghts,

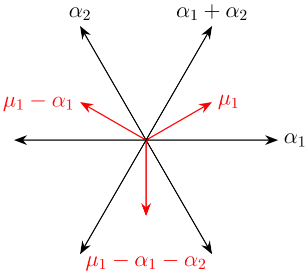
$$\vec{\mu}_1 = (\mu_1^1, \mu_1^2) \quad \text{and} \quad \vec{\mu}_2 = (\mu_2^1, \mu_2^2),$$

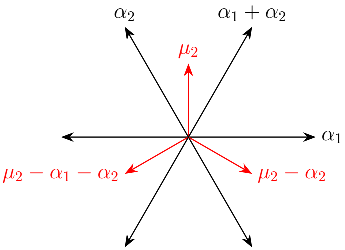
$$\mu_1 = \sqrt{2}\left(\frac{1}{2}, \frac{\sqrt{3}}{6}\right), \qquad \mu_2 = \sqrt{2}\left(0, \frac{2\sqrt{3}}{6}\right),$$

$$\mu_1 - \alpha_1 = \sqrt{2}\left(-\frac{1}{2}, \frac{\sqrt{3}}{6}\right), \qquad \mu_2 - \alpha_2 = \sqrt{2}\left(\frac{1}{2}, -\frac{\sqrt{3}}{6}\right),$$

$$\mu_1 - \alpha_1 - \alpha_2 = \sqrt{2}\left(0, -\frac{2\sqrt{3}}{6}\right), \qquad \mu_2 - \alpha_1 - \alpha_2 = \sqrt{2}\left(-\frac{1}{2}, -\frac{\sqrt{3}}{6}\right).$$

where $\vec{\mu}_1 = (\mu_1^1, \mu_1^2)$ and $\vec{\mu}_2 = (\mu_2^1, \mu_2^2)$ are 2-dim vectors, s. t.,





- Define spin, $T_3 = \frac{1}{\sqrt{2}} h_1$ and hypercharge, $Y = \frac{2}{\sqrt{6}} h_2$,

Quark	T_3	Y
$\|u\rangle = \|\mu_1\rangle$	$+\frac{1}{2}$	$+\frac{1}{3}$
$\|d\rangle = \|\mu_1 - \alpha_1\rangle$	$-\frac{1}{2}$	$+\frac{1}{3}$
$\|s\rangle = \|\mu_1 - \alpha_1 - \alpha_2\rangle$	0	$-\frac{2}{3}$
$\|\bar{u}\rangle = \|\mu_2 - \alpha_1 - \alpha_2\rangle$	$-\frac{1}{2}$	$-\frac{1}{3}$
$\|\bar{d}\rangle = \|\mu_2 - \alpha_2\rangle$	$+\frac{1}{2}$	$-\frac{1}{3}$
$\|\bar{s}\rangle = \|\mu_2\rangle$	0	$+\frac{2}{3}$

- Finite Dimensional algebras

$$[T^a,\ T^b] = i\ f_c^{ab}\ T^c, \qquad T^a \in \mathcal{G}, \quad a, b, c, = 1, \cdots dim\, \mathcal{G}$$

General Structure of Lie Algebras. $\mathcal{G} = \mathcal{H} \oplus \mathcal{P}$

- Observables $\{h_1, h_2, \cdots, h_r \in \mathcal{H}, \quad r = rank\}$, e.g., $J_3 = h_1$

$$[h_i,\ h_j] = 0, \qquad i, j = 1, \cdots, r$$

- Step operators, e.g., $J_{\pm} = E_{\pm\alpha} \in \mathcal{P}$, such that

$$[h_i,\ E_{\pm\alpha}] = \pm\alpha^i E_{\pm\alpha}, \qquad \vec{\alpha} = (\alpha^1, \alpha^2, \cdots, \alpha^r)$$

$\vec{\alpha}$ is a root, (**quanta**) and

$$[E_\alpha,\ E_\beta] = \begin{cases} \epsilon(\alpha,\beta)E_{\alpha+\beta}, & \alpha+\beta = \text{ root} \\ 2\frac{\sum \alpha^i h_i}{\alpha^2}, & \alpha+\beta = 0 \\ 0, \ \text{otherwise} \end{cases}$$

- Infinite Dimensional algebras (next lectures)

$$T_a \longrightarrow T^a_m, \quad m \in \mathbb{Z}$$

- Affine centrally extended Algebras (Kac-Moody)

$$[T^a_m,\ T^b_n] = i\ f^{ab}_c\ T^c_{m+n}\ \ + Km\,\delta_{m+n,0}\delta^{a,b}$$

- K describe central term (of QM origin),

$$[K, T^a_m] = 0$$

Roots lives in Non-Euclidian space.