Operator Algebras and the Formulation of Quantum Field Theory

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Brasília

08-10 of setember, 2025

General Description of a Physical Theory

Every physical theory should be able to specify

- Physically observable quantities and their mathematical description, the relationships between these observables, such as compatibility relations, algebraic relations, etc.
- The set of possible outcomes of individual measurements of these observables.
- The association between physical systems, observables, and the probability distributions that describe measurements of these observables in states.
- The set of pure states.
- The dynamics of observables and states.
- The symmetries of the described physical systems and their implementations in states and observables.

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These questions are not necessarily independent. The dynamics, for example, can be associated with symmetry by temporal evolution, and the set of pure states can be fixed by the algebra of observables.

• Observables, measurements and probability distributions

Each physical theory has its own set of observable quantities. Let A be an observable physical quantity and $\mathcal{C}(A)$ be the set of possible values resulting from measurements of A (in any state).

It is an experimental fact that repeated measurements of an observable A, maintained under the same conditions, that is, in the same physical state E of the system under study, do not necessarily yield the same value in $\mathcal{C}(A)$, having a random character.

It is an observational fact that an ideally infinite succession of experimental measurements of A, *all under the same physical conditions of the system in question*, should produce a statistical distribution in $\mathcal{C}(A)$ defined by a probability measure.

Let us denote the probability measure in question by $\mu_{E,\,A}$.

This probability measure $\mu_{E,\,A}$ is a function of both the set of conditions E that specifies the system and the observable A under consideration. This probability measure $\mu_{E,\,A}$ is called the *state* (or *physical state*) of the system in question with respect to the observable A.

The probability measure $\mu_{E,A}$, that is, the physical state of the system, contains within itself all available information about these properties.

There are three possible origins for the randomness mentioned above, observed in the measurement of an observable in a physical system. These origins can occur concomitantly:

- it can arise from experimental measurement errors,
- it can arise from incomplete knowledge of the system studied, or
- it can be intrinsic to the system described, a fact first identified in Atomic Physics.

Typically, when developing physical theories, the ideal situation is considered in which experimental inaccuracies are neglected. However, these still remain the two other sources of randomness, which must then be duly considered in the theoretical slidework.

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• Mean value, variance and correlations

In the statistical analysis of the measurement results of an observable A of a physical system in a given state, several quantities play a role.

One of them is the so-called *mean value*, or *expected value*, which will be denoted here by

$$\langle A \rangle_E = \int_{\mathcal{C}(A)} \lambda \, d\mu_{E,A}(\lambda) .$$

Other relevant quantities are the momenta

$$\langle A^n \rangle_E = \int_{\mathcal{C}(A)} \lambda^n \, d\mu_{E,A}(\lambda) ,$$

$n \in \mathbb{N}$.

It is a well-known mathematical fact – a consequence of Weierstrass Theorem (*"Hamburger momentum problem"*) – that if $\mathcal{C}(A)$ is a compact set, then the probability measure $\mu_{E,A}$ can be recovered from the set of all momenta $\langle A^n \rangle_E$, $n \in \mathbb{N}$.

Another important stochastic quantity is the so-called *variance*, defined by

$$\mathrm{Var}_E(A) \;:=\; \langle A^2\rangle_E - \langle A\rangle_E^2 \;=\; \left\langle \left(A - \langle A\rangle_E\right)^2\right\rangle_E \;\geq\; 0\;, \tag{1}$$

which provides a qualitative indication of how much the value of the variation of A deviate from their mean value.

Although it is not the only stochastic quantity that provides this type of qualitative information, variance is a useful quantity: Heisenberg's famous *Uncertainty Relations* in Quantum Mechanics are statements about the variance of two observables that do not commute (for example, momentum and position in the same Cartesian direction: $\mathrm{Var}(p_x)\,\mathrm{Var}(x) \geq \hbar^2/4$).

The *correlation*, or *covariance*, between two observables A and B, relative to a state E is defined by

$$\mathrm{Cov}_E(A,\ B)\ :=\ \Big\langle \big(A - \langle A\rangle_E\big)\big(B - \langle B\rangle_E\big)\Big\rangle_E \tag{2}$$

and, as one easily sees,

$$\mathrm{Cov}_E(A,\ B)\ =\ \big\langle AB\big\rangle_E - \langle A\rangle_E\langle B\rangle_E\ .$$

In words, $\mathrm{Cov}_E(A,\ B)$ "measures" how much the average departure of A from its mean value $\langle A\rangle_E$ is statistically related to the average departure of B from its mean value $\langle B\rangle_E$.

If A and B are stochastically independent, then $\mathrm{Cov}_E(A,\ B) = 0$. The converse is not generally true.

In Probability Theory, the expected value (or "expectancy") of a measurable function ("random variable") A defined on a sample space Ω and its variance with respect to a probability measure μ on Ω are given by

$$
\begin{aligned}
\mathbb{E}_\mu(A) \quad &\equiv \quad \langle A \rangle_\mu \; := \int_\Omega A \, d\mu \; , \\
\mathrm{Var}_\mu(A) \quad &:= \quad \int_\Omega \left(A - \langle A \rangle_\mu\right)^2 d\mu \; = \; \mathbb{E}_\mu\left(A^2\right) - \mathbb{E}_\mu\left(A\right)^2 \; ,
\end{aligned}
$$

The correlation, or covariance, of two random variables A and B defined in a sample space Ω, with respect to a probability measure μ in Ω is given by

$$
\begin{aligned}
\mathrm{Cov}_E(A,\; B) \; &= \; \int_\Omega \Big(\left(A - \langle A \rangle_\mu\right)\left(B - \langle B \rangle_\mu\right)\Big) d\mu \\
&= \; \int_\Omega AB \; d\mu - \left(\int_\Omega A \; d\mu\right)\left(\int_\Omega B \; d\mu\right) \\
&= \; \mathbb{E}_\mu\left(AB\right) - \mathbb{E}_\mu\left(A\right)\mathbb{E}_\mu\left(B\right) \; .
\end{aligned}
$$

• Variance and pure states

In probability theory, a probability measure on a sample space μ is said to be *pure* if it cannot be written as a convex linear combination of two other distinct probability measures of μ from the same sample space, that is, if it cannot be written in the form $\mu = \alpha\mu_1 + (1-\alpha)\mu_2$ where μ_1 and μ_2 are distinct probability measures and $0 < \alpha < 1$. It is an easy exercise to show that if $\mu = \alpha\mu_1 + (1-\alpha)\mu_2$, then

$$\langle A\rangle_\mu = \alpha\langle A\rangle_{\mu_1} + (1-\alpha)\langle A\rangle_{\mu_2}$$

and

$$\mathrm{Var}_\mu(A) = \alpha\mathrm{Var}_{\mu_1}(A) + (1-\alpha)\mathrm{Var}_{\mu_2}(A) + \alpha(1-\alpha)\left[\langle A\rangle_{\mu_1} - \langle A\rangle_{\mu_2}\right]^2 .$$

Therefore,

$$\mathrm{Var}_\mu(A) \geq \alpha\mathrm{Var}_{\mu_1}(A) + (1-\alpha)\mathrm{Var}_{\mu_2}(A) \geq \min\left\{\mathrm{Var}_{\mu_1}(A), \mathrm{Var}_{\mu_2}(A)\right\} .$$

In this sense, pure probability measures represent those with the smallest possible deviation of the quantity represented by A from its mean value.

• Purity of states

We say that a physical system is in a *pure state* for any given observable A if $\mu_{E,A}$ is pure.

The pure states of a physical system thus represent those with the smallest “fluctuations” of the observable quantities A.

We thus understand that determining which states a physical system has and what the variances of observables in these pure states are provides important information about the smallest possible fluctuations that can be observed in that system.

This is important information about the degree of intrinsic randomness (i.e., not arising from experimental errors or incomplete knowledge) of the underlying physical theory that describes the system in question.

• The Picture of Classical Mechanics

In classical mechanics, observables are functions in phase space and states are probability distributions in phase space.

$$\langle f\rangle \;=\; \int_{\mathbb{F}} f(q,\,p)\,\rho(q,\,p)\;dqdp\;,$$

com $\rho(q,\,p) \geq 0$ e $\displaystyle\int_{\mathbb{F}} \rho(q,\,p)\;dqdp = 1.$

Pure states are given by Dirac measures:

$$\langle f\rangle \;=\; \int_{\mathbb{F}} f(q,\,p)\,\delta(p-p_0,\;q-q_0)\;dqdp \;=\; f(q_0,\,p_0)\;.$$

Time evolution (in $L^2(\mathbb{F}, dqdp)$):

$$\frac{d}{dt} f(q_t, p_t) = (\mathcal{L} f)(q_t, p_t) ,$$

where

$$\mathcal{L} := \frac{\partial \mathcal{H}}{\partial p} \frac{\partial}{\partial q} - \frac{\partial \mathcal{H}}{\partial q} \frac{\partial}{\partial p} ,$$

the Liouville operator.

Quantum Physics

- **The Spectral Theorem and probability distributions in the spectrum**

If $\psi \in \mathcal{H}$ ia a non zero vector in a (separable) Hilbert space $\mathcal{H}$ and a bounded selfadjoint operator A acting on $\mathcal{H}$, we know by the Spectral Theorem that

$$\langle \psi, \, A\psi \rangle \; = \int_{\sigma(A)} \lambda \, d\mu_{\psi, A}(\lambda) \; = \int_{\sigma(A)} \lambda \, d\langle \psi, \, P_\lambda \psi \rangle \; .$$

$\mu_{\psi, A}$ is a positive measure on $\sigma(A)$ and, if $\|\psi\| = 1$, one has

$$\int_{\sigma(A)} d\mu_{\psi, A} \; = \int_{\sigma(A)} d\langle \psi, \, P_\lambda \psi \rangle \; = \; 1 \; .$$

Hence, $\mu_{\psi, A}$ is a probability measure on $\sigma(A)$.

• The Picture of Quantum Mechanics

Basic postulates of Quantum Mechanics:

- Observables are represented by self-adjoint operators acting on a separable Hilbert space (*e.g.*, $L^2(\mathbb{R}, dx)$).
- Individual measuremente of an observable A always produce elements of $\sigma(A)$, the spectrum of A.
- The physical states of a quantum system with a finite number of degrees of freedom are described by "density matrices" acting on a Hilbert space $\mathcal{H}$, i.e., positive self-adjoint operators ρ with $\mathrm{Tr}\,(\rho) = 1$ such that the mean value of an ideally infinite set of measurements of the observable A in the state described by ρ is given by

$$\langle A \rangle = \mathrm{Tr}\,(\rho A)\ .$$

- Pure states correspond to one-dimensional projections: $P \equiv |\psi\rangle\langle\psi|$, for $\psi \in \mathcal{H}$ with $\|\psi\| = 1$.

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The choice of self-adjoint operators for the observables is motivated by two properties:

1. the spectrum of a self-adjoint operator is always a subset of the real line, a fact consistent with the postulate that individual measurements of an observable must be elements of the spectrum of the associated operator;
2. the spectral theorem states that self-adjoint operators can be represented by sums (or integrals) of the type $A = \sum_{\lambda \in \sigma(A)} \lambda P_\lambda$. Here, P_λ designates the projector over the eigenspace of A with eigenvalue λ. $\sigma(A)$ denotes the spectrum of A.
   (For continuum spectrum the sum symbol used above has only a formal meaning and should be replaced by an integral symbol $A = \int_{\sigma(A)} \lambda \, dP_\lambda$, in the sense described in the Spectral Theorem).

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Hence, for a state described by a density matrix ρ,

$$\langle A\rangle \;=\; \mathrm{Tr}(\rho A) \;=\; \mathrm{Tr}\left(\rho \sum_{\lambda\in\sigma(A)} \lambda\, P_\lambda\right) \;=\; \sum_{\lambda\in\sigma(A)} \lambda\, \mathrm{Tr}\left(\rho P_\lambda\right) \;=\; \sum_{\lambda\in\sigma(A)} \lambda p_\lambda \;,$$

where

$$p_\lambda \;:=\; \mathrm{Tr}\left(\rho P_\lambda\right)$$

satisfies

$$p_\lambda \;\geq\; 0 \qquad \text{and} \qquad \sum_{\lambda\in\sigma(A)} p_\lambda \;=\; 1\;,$$

and, therefore, can be interpreted as a probability distribution on $\sigma(A)$, the set of all possible measurement values of the observable A.

There is much more to be said, but a very important point is that if A and B represent two observables that commute:

$$AB = BA ,$$

then they are *compatible*: their measurements can be performed independently.

• The algebraic structure of QM

For $\psi \in L^2(\mathbb{R}, dx)$,

$$(P\psi)(x) := -i\hbar \frac{d\psi}{dx}(x) , \qquad \text{and} \qquad (Q\psi)(x) = x\phi(x) .$$

They satisfy Heisenberg commutation relations:

$$PQ - QP = -i\hbar .$$

P and Q cannot be defined everywhere: take

$$\psi(x) := \begin{cases} 0 , & \text{for } x < 1 , \\ \frac{1}{x} , & \text{for } x \geq 1 . \end{cases}$$

This is a vector in $L^2(\mathbb{R}, dx)$, but

$$(Q\psi)(x) := \begin{cases} 0 , & \text{for } x < 1 , \\ 1 , & \text{for } x \geq 1 , \end{cases}$$

is **not** a vector in $L^2(\mathbb{R}, dx)$.

A solution is to replace then by the Weyl operators (here we take $\hbar = 1$):

$$\begin{aligned} \big(U(a)f\big)(x) &:= f(x-a)\ , \\ \big(V(a)f\big)(x) &:= e^{ia\,x} f(x)\ , \end{aligned}$$

$a \in \mathbb{R}$. One has,

$$\begin{aligned} U(a) &= \exp\big(-iaP\big)\ , \\ V(a) &= \exp\big(iaQ\big)\ , \end{aligned}$$

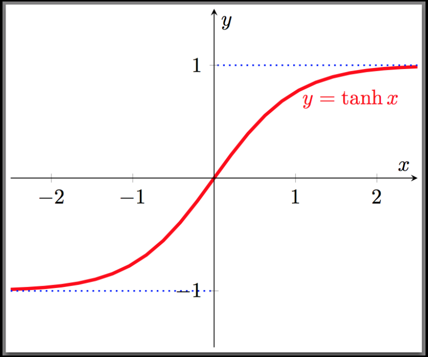
and the Weyl relations:

$$\begin{aligned} U(a)V(b) &= e^{-ia\,b} V(b)U(a)\ , \\ U(a)U(a') &= U(a+a') = U(a')U(a)\ , \\ V(b)V(b') &= V(b+b') = V(b')V(b)\ , \end{aligned}$$

$a,\ b \in \mathbb{R}$.

The general philosophy is that we can always replace observables by bounded observables.

Example: instead of measuring positions with the multiplication operator "x" we can measure "$\tanh(x)$", a bounded and one-to-one function in $\mathbb{R}$.



This leads us to the important definition of a bounded operator on a Hilbert space:

$$\|A\| \;:=\; \sup\left\{ \frac{\|A\psi\|}{\|\psi\|},\; \psi \in \mathcal{H},\; \psi \neq 0 \right\} \;,$$

with $\|\psi\|^2 := \langle \psi,\, \psi \rangle$.

If this quantity is finite, A is said to be a **bounded operator**.

The set of all bounded operators acting on $\mathcal{H}$ is an algebra, denoted by $\mathcal{B}(\mathcal{H})$. One has, for all $A,\, B \in \mathcal{B}(\mathcal{H})$, Let us recall that a Hilbert space is provided with a scalar product $\langle\psi,\ \phi\rangle$ (denoted by $\langle\psi|\phi\rangle$ in physicists texts) with $\|\psi\|^2 := \langle\psi|\psi\rangle$ and for any $A \in \mathcal{B}(\mathcal{H})$ we can define another operator A^* (denoted by $A^\dagger$ in physicists texts) so that

$$\left\langle\psi,\ A\phi\right\rangle \ = \ \left\langle A^*\psi,\ \phi\right\rangle$$

for all $\psi,\ \phi \in \mathcal{H}$. The map $A \mapsto A^*$ is

- antilinear: $\left(\alpha A + \beta B\right)^* = \overline{\alpha}A^* + \overline{\beta}B^*$;
- idempotent: $\left(A^*\right)^* = A$.
- anti-homomorphic: $(AB)^* = B^*A^*$.

An operator is said to be **selfadjoint** if $A = A^*$.

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For bounded operators ons has

$$\begin{aligned} \|AB\| &\leq \|A\|\,\|B\| \,, \\ \|A^*\| &= \|A\| \,, \\ \|A^*A\| &= \|A\|^2 \,, \end{aligned}$$

Bounded operators have a bounded (compact) spectrum.

Bounded operators can be defined everywhere.

- **Uncertainty relations**

The variance of an observable B in a state ω (for instance $\omega(B) = \mathrm{Tr}(\rho B)$) is defined by

$$\mathrm{Var}_\omega(B) \;:=\; \left\langle \left(B - \langle B\rangle_\omega\right)^2 \right\rangle_\omega \;=\; \left\langle B^2\right\rangle_\omega - \left\langle B\right\rangle_\omega^2 \;=\; \omega\left(B^2\right) - \omega(B)^2$$

and the covariance of two observables A and B is given by

$$\begin{aligned}\mathrm{Cov}_\omega(A,\;B) \;:=\; & \frac{1}{2}\omega\Big(\big(A - \omega(A)\big)\big(B - \omega(B)\big) + \big(B - \omega(B)\big)\big(A - \omega(A)\big)\Big) \\ =\; & \frac{1}{2}\omega\big(AB + BA\big) - \omega(A)\omega(B)\;.\end{aligned}$$

One has *Heisenberg-Robertson-(Kennard-Weyl-Pauli) uncertainty relation*:

$$\mathrm{Var}_\omega(A)\mathrm{Var}_\omega(B) \;\geq\; \frac{1}{4}\omega\Big(i[A,\,B]\Big)^2 \;.$$

For instance,

$$\mathrm{Var}_\omega(P)\mathrm{Var}_\omega(Q) \;\geq \frac{\hbar^2}{4} \;.$$

Moreover, one has *Schrödinger's uncertainty relation*:

$$\mathrm{Var}_\omega(A)\mathrm{Var}_\omega(B) \;\geq\; \mathrm{Cov}_\omega(A,\;B)^2 + \frac{1}{4}\omega\Big(i[A,\,B]\Big)^2 \;.$$

• The mathematical problems of Quantum Field Theory

- Divergences in "naive" perturbation theory $\longrightarrow$ Regularization/renormalization.
- Problems with perturbation theory: even after renormalization, perturbative series do not seem to converge! They seem to behave like

$$\sum_{n=0}^{\infty} n!\, g^n \ .$$

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- Conceptual problems: what do these theories describe? Fields? Particles? Superselection sectors?
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• Wightman theories

Main ingredients: Wightman tempered distributions

$$W_n(x_1, \ldots, x_n) \equiv \langle \Omega, \ \Phi(x_1) \cdots \Phi(x_n)\, \Omega \rangle$$

$+$ positivity, Poincaré covariance, **Einstein causality** etc.

Einstein causality means in this context – bosonic case:

$$W_2(x_1, x_2) = W_2(x_2, x_1)$$

etc, for space-like separated points.

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Most important results:

- Wightman's reconstruction theorem: Hilbert Space, (unbounded) operator for the fields.
- PCT Theorem.
- Spin and Statistics.
- Reeh-Schlieder theorem.
- Bisognano-Wichman theorem.
- Haag-Ruelle scattering theory.
- Euclidean version: Schwinger functions and the construction of Euclidian models.

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Wightman's reconstruction theorem allows the association

$$f \longmapsto \Phi(f) \equiv \int f(x)\Phi(x)\,dx$$

as an (unbounded) operator acting on the Hilbert space (operator valued distributions). f is taken as a function in Schwartz space.

Einstein locality implies

$$\Phi(f)\Phi(g) = \Phi(g)\Phi(f)$$

if $\operatorname{supp}(f)$ and $\operatorname{supp}(g)$ are space-like.

The Poincaré group is implemented in the Hilbert space by unitary transformations.

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The Algebraic Formulation of Quantum Fields

Rudolf Haag, Daniel Kastler, Huzihiro Araki,

Sergio Doplicher, John Roberts, Bert Schroer,

Detlev Buchholz, Klaus Fredenhagen.

• C*-algebras. What are they and why they are useful for Physics

A C*-algebra is a normed, complete, associative algebra over $\mathbb{C}$ with an antilinear involution $A \mapsto A^*$, such that

- $\|AB\| \leq \|A\| \, \|B\|$,
- $\|A^*\| = \|A\|$,
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for all A, B elements of the algebra.

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• Spectrum of an element of a C*-algebra

If $\mathcal{A}$ is a C*-algebra with unit, we say that $\lambda \in \mathbb{C}$ is an element of the spectrum of A if

$$(\lambda \mathbb{1} - A)^{-1}$$

does not exist in $\mathcal{A}$.

The spectrum of $A \in \mathcal{A}$ is denoted by $\sigma(A)$.

$\sigma(A)$ is always closed and bounded and $\sigma(A) \subset \mathbb{R}$ if A is selfadjoint.

For A self-adjoint

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A C^*-algebra may or may not have a unit $\mathbb{1}$. In physical applications we always assume there is one.

C^*-algebras are abstract algebras and do not necessarily act on vector spaces. However, they can be represented as operator algebras acting on Hilbert spaces.

Moreover, C^*-algebras also admit a probabilistic interpretation for expectarion values. To understand that we have to introduce the important notion os *state*.

Exemples of C^*-algebra are:

- $\mathcal{B}(\mathcal{H})$,
- The algebra of compact operators acting on Hilbert spaces,
- CAR and CCR algebras,
- AF-algebras,
- Cuntz algebras and Cuntz-Krieger algebras,

and many more.

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- linear: $\omega(\alpha A + \beta B) = \alpha\omega(A) + \beta\omega(B)$.
- positive: $\omega(A^*A) \geq 0$.
- normalization: $\omega(\mathbb{1}) = 1$.

It follows that

- $\omega(A^*) = \overline{\omega(A)}$.
- ω is continuous: if $\|A_n - A\| \overset{n\to\infty}{\longrightarrow} 0$, then $\lim_{n\to\infty} \omega(A_n) = \omega(A)$.
- For each selfadjoint $A \in \mathcal{A}$ there is a probability measure $\mu_{A,\omega}$ on $\sigma(A)$ such that

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Hence, states can be interpreted as expectation values.

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• The GNS construction

Theorem [GNS Representation]

Let ω be a state of an algebra $\mathbf{C}^*$ *which we will denote by $\mathcal{A}$. It is possible to construct a Hilbert space $\mathcal{H}_\omega$ and a representation π_ω of $\mathcal{A}$ by bounded operators acting on $\mathcal{H}_\omega$ such that*

$$\pi_\omega(A^*) \;=\; \pi_\omega(A)^*$$

for all $A \in \mathcal{A}$ (a representation with this property is said to be a $*$-representation*). Furthermore, if the algebra $\mathcal{A}$ has a unit, then there exists in $\mathcal{H}_\omega$ a vector Ω with the property that*

$$\omega(A) \;=\; \langle \Omega,\; \pi_\omega(A)\Omega \rangle_{\mathcal{H}_\omega} \;.$$

This vector Ω is a cyclic vector for the representation π_ω, that is, $\{\pi_\omega(A)\Omega,\; A \in \mathcal{A}\}$ is a dense set in $\mathcal{H}_\omega$.

By a representation of the algebra I mean

$$\pi_\omega(\alpha A + \beta B) \;=\; \alpha\pi_\omega(A) + \beta\pi_\omega(B)$$

and

$$\pi_\omega(AB) \;=\; \pi_\omega(A)\pi_\omega(B) \;,$$

$\forall A, B \in \mathcal{A}, \forall \alpha, \beta \in \mathbb{C}$

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This vector Ω is a cyclic vector for the representation π_ω, that is, $\{\pi_\omega(A)\Omega,\; A \in \mathcal{A}\}$ is a dense set in $\mathcal{H}_\omega$.

By a representation of the algebra I mean

$$\pi_\omega\big(\alpha A + \beta B\big) \;=\; \alpha\pi_\omega(A) + \beta\pi_\omega(B)$$

and

$$\pi_\omega(AB) \;=\; \pi_\omega(A)\pi_\omega(B) \;,$$

$\forall A,\, B \in \mathcal{A},\, \forall \alpha,\; \beta \in \mathbb{C}$

• Simplified proof

Considere $\mathcal{A}$ as a vector space and identify $\Omega \equiv \mathbb{1}$.

Define $\pi_\omega(A)\Omega := A\mathbb{1} = A$.

For two vetors $A,\ B \in \mathcal{A}$, define a scalar product $\langle A,\ B\rangle_\omega := \omega(A^*B)$, that means

$$\langle \pi_\omega(A)\Omega,\ \pi_\omega(B)\Omega\rangle_\omega \ = \ \omega(A^*B)\ .$$

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$$\langle \Omega,\ \pi_\omega(B)\Omega\rangle_\omega \ = \ \omega(B)\ .$$

By these definitions, one has $\left\|\pi_\omega(A)\Omega\right\|^2 = \omega(A^*A)$.

Now, complete the set $\left\{\pi_\omega(A)\Omega,\ A \in \mathcal{A}\right\}$ in this norm, producing a Hilbert space $\mathcal{H}_\omega$.

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$$\left\|\pi_\omega(A)\pi_\omega(B)\Omega\right\|_\omega^2 \;=\; \left\langle \pi_\omega(A)\pi_\omega(B)\Omega,\; \pi_\omega(A)\pi_\omega(B)\Omega \right\rangle_\omega \;=\; \omega\big(B^*A^*AB\big)$$
$$\leq\; \|A^*A\|\,\omega\big(B^*B\big) \;=\; \|A^*A\|\,\left\|\pi_\omega(B)\Omega\right\|_\omega^2 \,,$$

and, therefore

$$\|\pi_\omega(A)\|^2 \;\leq\; \|A^*A\| \;=\; \|A\|^2 \,,$$

showing that $\pi_\omega(A)$ are bounded operators acting on $\mathcal{H}_\omega$.

Finnaly, for $A,\ B,\ C \in \mathcal{A}$ one has

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and, hence, $\pi_\omega(A)\pi_\omega(B) = \pi_\omega(AB)$, since $\left\{\pi_\omega(C)\Omega,\ C \in \mathcal{A}\right\}$ is a dense set in $\mathcal{H}_\omega$.

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• The GNS construction and the purity of states

If ω is a pure state, π_ω is an irreductible representation!

What happens if ω is not a pure state?

Take, for istance, $\omega = \lambda\omega_1 + (1-\lambda)\omega_2$, with ω_1 and ω_2 pure. Then,

$$\pi_\omega(A) = \begin{pmatrix} \pi_{\omega_1}(A) & 0 \\ 0 & \pi_{\omega_2}(A) \end{pmatrix},$$

that means $\pi_\omega = \pi_{\omega_1} \oplus \pi_{\omega_2}$ and $\mathcal{H}_\omega$ also splits into two orthogonal subspaces where π_{ω_1} and π_{ω_2} act irreducibly.

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• von Neumann-algebras

What are they and why they are usefull for Physics.

Let $\mathcal{A}$ be a C*-subalgebra of some $\mathcal{B}(\mathcal{H})$. We denote by $\mathcal{A}'$ the commutant of $\mathcal{A}$:

$$\mathcal{A}' := \left\{ B \in \mathcal{B}(\mathcal{H}) \middle| BA = AB \text{ for all } A \in \mathcal{A} \right\} .$$

By definition, one has

$$\mathcal{A} \subset \mathcal{A}'' .$$

A C*-subalgebra $\mathcal{N}$ of some $\mathcal{B}(\mathcal{H})$ is say to be a *von Neumann algebra* if

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Equivalently (by von Neumann's bicommutant theorem), $\mathcal{N}$ is a von Neumann algebra if it is *weakly closed*: if

$$\lim_{n\to\infty} \langle \psi, A_n \phi \rangle = \langle \psi, A\phi \rangle$$

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• The Algebraic Formulation of QFT

The Haag-Kastler postulates.

Why von Neumann algebras? (Einstein causality).

There is a net of observable C*-algebras $\mathcal{O} \to \mathcal{A}(\mathcal{O})$ ($\mathcal{O}$ open subsets of Minkowski space with compact closure).

- Isotony: $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$ if $\mathcal{O}_1 \subset \mathcal{O}_2$.
  This allows to define the C*-algebra

$$\mathfrak{A} := \bigcup_{\mathcal{O}} \mathcal{A}(\mathcal{O})$$

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  This allows to consider the algebras $\mathcal{A}(\mathcal{O})$ as von Neumann algebras.
- Poincaré covariance: for g in the Poincaré group

$$\mathcal{A}(g\mathcal{O}) = U(g)^* \mathcal{A}(\mathcal{O}) U(g) ,$$

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- Spectrum condition: The joint spectrum of the generators of translations is contained in the closed forward light cone.
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Reeh-Schlieder Theorem

- **Weak additivity**

Consider, as before, the inductive limit of C*-algebras

$$\mathfrak{A} := \bigcup_{\mathcal{O}} \mathcal{A}(\mathcal{O})$$

We say that *weak additivity* holds for a state φ if

$$\pi_\varphi(\mathbb{A})'' = \left(\bigcup_{x \in \mathbb{M}} \pi_\varphi\Big(\mathcal{A}(\mathcal{O} + x)\Big) \right)'' .$$

• Cyclic and separating vectors

Let $\mathcal{N}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$. A vector $\Omega \in \mathcal{H}$ is said to be

- **cyclic** for $\mathcal{N}$ if the set of vectors $\{A\Omega,\ A \in \mathcal{N}\}$ is dense in $\mathcal{H}$.
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Reeh-Schlieder Theorem (1961)

Assume weak additivity for the vacuum state ω. Then, the vacuum vector Ω obtained from the GNS construction from ω is

- a **cyclic** vetor for $\pi_\omega\big(\mathcal{A}(D)\big)$ for any open domain D.
- a **cyclic** and **separating** vetor for $\pi_\omega\big(\mathcal{A}(D)\big)$ for any open domain D with $D' \neq \emptyset$.

Meaning and discussion.

Intuitive or counter-intuitive result?

The notion of total set in a Hilbert space.

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• Tomita-Takesaki Theorem

Let $\mathcal{N}$ be a von Neumann algebra and Ω be a cyclic and separating vector for $\mathcal{N}$. Then,

- There is a antilinear operator J such that

$$J\mathcal{N}J = \mathcal{N}' .$$

  $J^2 = \mathbb{1}$.

- These is a positive and self-adjoint operator Δ such that

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  for all $t \in \mathbb{R}$.

- Ω is a KMS state (a temperature state) with $\beta = 1/2$ for the dynamics defined in $\mathcal{N}$ by Δ^{it}, $t \in \mathbb{R}$.

• Tomita-Takesaki Theorem

Let $\mathcal{N}$ be a von Neumann algebra and Ω be a cyclic and separating vector for $\mathcal{N}$. Then,

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- **Bisognano-Wichman Theorem**

Unruh effect.

Comment of hyperfinite factors of type III_1.

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- Huzihiro Araki. *Mathematical Theory of Quantum Fields*. Oxford, (1999).
- Rudolf Haag. *Local Quantum Physics*. Springer Verlag (1996).
- R. F. Streater and A. S. Wightman. *PCT, Statistics and All That.* Princeton Univ. Press. (1980).
- R. Jost. *The General Theory of Quantized Fields*. American Mathematical Society (1965).
- N. N. Bogoliubov, A. A. Logunov, A. I. Oksak and I. T. Todorov. *General Principles of Quantum Field Theory*. Kluwer (1990).
- Michael Reed and Barry Simon. *Methods of Modern Mathematical Physics. Vol. 1-4*. Academic Press. New York. (1972–1979).
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